International Journal of Theoretical Physics, Vol. 37, No. 9, 1998

Two-State Paramagnetism Induced by Tsallis and Renyi Statistics

C. Wolf[1]

Received December 11, 1997

We calculate the induced magnetization of a two-state system within the context of Tsallis statistics and Renyi statistics. Our calculation demonstrates that the magnetization is increased by Tsallis statistics and decreased by Renyi statistics relative to the Boltzmann Gibbs value.

1. INTRODUCTION

In the past decade a new approach to statistical mechanics has developed motivated by the study of multifractals (Tsallis, 1988) along with studies in coarse graining to represent a local entropy (Renyi, 1970). The above "q statistics" applies to systems with long-range interactions and a non-Markovian memory (Tirnakli *et al.*, 1997a) and naturally includes systems admitting gravitational forces (Pavon, 1987), magnetic systems (Hiley and Joyce, 1965), and processes involving anomalous diffusion (Montroll and Schlesinger, 1983). In particular, "Tsallis statistics" when applied to the solar plasma predicts the correct rate of solar neutrino production in accord with solar neutrino data (Clayton, 1974; Bahcall and Pinsonneault, 1992; Kaniadakis *et al.*, 1996). Other applications of Tsallis statistics include studies in the "generalized H theorem" (Ramshaw, 1993a,b; Mariz, 1992; Plastino and Plastino, 1993), the fluctuation-dissipation theorem (Chame and de Mello, 1994), the Langevin and Fokker–Planck equations (Stariolo, 1994), the equipatation theorem (Plastino *et al.*, 1994), the Ising chain (Andrade, 1991, 1994), and the problem of blackbody radiation (Tsallis *et al.*, 1995; Tirnakli *et al.*, 1997b). Few applications of Renyi's statistics are found in the literature

[1] Department of Physics, Massachusetts College of Liberal Arts, North Adams, Massachusetts 01247.

0020-7748/98/1000-243 3$15.00/0

primarily because of its rather pure mathematical foundations. In the present paper we compare the Tsallis statistics with the Renyi statistics when applied to a simple paramagnetic two-state system. We find that Tsallis statistics generates a positive correction to the Boltzmann Gibbs value of the magnetization, while the Renyi statistics generates a negative correction. Both of these schemes reduce to the Maxwell–Boltzmann value for the magnetization when the q parameter approaches 1.

2. PARAMAGNETISM WITHIN TSALLIS AND RENYI STATISTICS

We begin by writing the expression for the entropy within the context of Tsallis statistics (for a single particle)

$$S_1 = \frac{k}{q-1}\left(1 - \sum P_i^q\right)$$

$P_i = N_i/N$, k is the Boltzmann constant, and q is the q parameter. For N particles we have

$$S = \frac{kN}{q-1}\left(1 - \sum P_i^q\right) \tag{2.1}$$

$(q \neq 1)$.

If S is varied with respect to N_i with the constraints

$$\sum N_i = N, \qquad \sum N_i \varepsilon_i = N$$

we obtain (Tirnakli *et al.*, 1997a)

$$P_i = \frac{N_i}{N} = \frac{(1 - \beta(q-1)\varepsilon_i)^{1/(q-1)}}{Z} \tag{2.2}$$

($\beta = 1/kT$, with k the Boltzmann constant), where $Z = \Sigma_i\,[1 - \beta(q - 1)\varepsilon_i]^{1/(q-1)}$.

We now consider the two-state spin system in a z-component magnetic field,

$$\varepsilon_+ = -\bar{\mu}B, \qquad \varepsilon_- = \bar{\mu}B$$

$\bar{\mu}$ is the magnetic moment; + represents $\bar{\mu}$ in the direction of the field, − represents $\bar{\mu}$ against the field. Equation (2.2) can be written as

$$\begin{aligned} P_i &= \frac{\exp\{[1/(q-1)]\ln_e[1 - \beta(q-1)\varepsilon_i]\}}{Z} \\ &\cong \frac{\exp[-\beta\varepsilon_i - \beta^2(q-1)\varepsilon_i^2/2 - \beta^3(q-1)^2\varepsilon_i^3/3]}{Z} \end{aligned} \tag{2.3}$$

Using $\varepsilon_+ = -\bar{\mu}B$ and $\varepsilon_- = \bar{\mu}B$, evaluating Z, and using $P_+ = N_+/N$ and $P_- = N_-/N$, we obtain for the magnetization of N spins, $M = N_+\bar{\mu} - N_-\bar{\mu}$, or

$$\begin{aligned} M = N\bar{\mu}\Bigg\{&\exp\left[\beta\bar{\mu}B - \frac{\beta^2(q-q)(\bar{\mu}B)^2}{2} + \frac{\beta^3(q-1)^2(\bar{\mu}B)^3}{3}\right] \\ &- \exp\left[-\beta\bar{\mu}B - \frac{\beta^2(q-1)(\bar{\mu}B)^2}{2} - \frac{\beta^3(\bar{\mu}B)^3(q-1)^2}{3}\right]\Bigg\} \\ &\times\Bigg\{\exp\left[\beta\bar{\mu}B - \frac{\beta^2(q-1)(\bar{\mu}B)^2}{2} + \frac{\beta^3(q-1)^2(\bar{\mu}B)^3}{3}\right] \\ &+ \exp\left[-\beta\bar{\mu}B - \frac{\beta^2(q-1)(\bar{\mu}B)^2}{2} - \frac{\beta^3(\bar{\mu}B)^3(q-1)^2}{3}\right]\Bigg\}^{-1} \end{aligned} \tag{2.4}$$

When we evaluate equation (2.4) to order $(q-1)^2$ we obtain

$$M = N\bar{\mu}\tanh(\beta\bar{\mu}B)\left(1 + \frac{(q-1)^2\beta^3(\bar{\mu}B)^3}{3\sinh(\beta\bar{\mu}B)\cosh(\beta\bar{\mu}B)}\right) \tag{2.5}$$

From equation (2.5) we see that the first correction term increases in proportion to $(q-1)^2$ and $(\bar{\mu}B/kT)^3$. For the Renyi statistics (Renyi, 1970) we have $S_1 = [k/(1-q)]\ln(\Sigma P_i^q)$; if we write S for N particles we have

$$S = \frac{kN}{1-q}\ln_e\sum\left(\frac{N_i}{N}\right)^q \tag{2.6}$$

We now have upon maximizing S with the constraints $\Sigma N_i = \text{const}$, $\Sigma N_i\varepsilon_i = \text{const}$

$$\frac{q}{1-q}\left(\frac{(N_i/N)^{q-1}}{\sum (N_i/N)^1}\right) = \frac{\varepsilon_i - \mu}{\tau} \tag{2.7}$$

μ is the chemical potential, $\tau = kT$.

Here μ/τ is the Lagrange multiplier for $\delta\,\Sigma N_i = 0$, and $-\frac{1}{\tau}$ is the Lagrange multiplier for $\delta\,\Sigma\,\varepsilon_i N_i = 0$.

For a two-level system with ε_1, ε_2 ($\varepsilon_1 > \varepsilon_2$) we have from equation (2.7)

$$\frac{N_1}{N_2} = \left(\frac{\varepsilon_1 - \mu}{\varepsilon_2 - \mu}\right)^{1/(q-1)} \tag{2.8}$$

Letting $N_1 = N - N_2$, we find

$$N_2 = \frac{N}{1 + ((\varepsilon_1 - \mu)/(\varepsilon_2 - \mu))^{1/(q-1)}},$$
$$N_1 = N\frac{((\varepsilon_1 - \mu)/(\varepsilon_2 - \mu))^{1/(q-1)}}{1 + ((\varepsilon_1 - \mu)/(\varepsilon_2 - \mu))^{1/(q-1)}} \tag{2.9}$$

Substituting N_1, N_2 back into equation (2.7) for $i = 1$, we find

$$\frac{q}{1-q}\left(\frac{1 + X^{1/(q-1)}}{1 + X^{q/(q-1)}}\right)X = \frac{\varepsilon_1 - \mu}{\tau} \tag{2.10}$$

$[X = (\varepsilon_1 - \mu)/(\varepsilon_2 - \mu)]$ for $\tau = kT > \overline{\mu}B$, $X \simeq 1$, and we find $\mu = \tau(q/(q-1)) + \varepsilon_1$ $(\varepsilon_1 = \overline{\mu}B,\ \varepsilon_2 = -\overline{\mu}B)$.

Substituting this value of μ into equation (2.9) we obtain

$$N_1 = \frac{N/(1 + (\varepsilon_1 - \varepsilon_2)(q-1)/\tau q)^{1/(q-1)}}{1 + 1/(1 + (\varepsilon_1 - \varepsilon_2)(q-1)/\tau q)^{1/(q-1)}}$$
$$N_2 = \frac{N}{1 + 1/(1 + (\varepsilon - \varepsilon_2)(q-1)/\tau q)^{1/(q-1)}} \tag{2.11}$$

If we approximate

$$\left(1 + \frac{(\varepsilon_1 - \varepsilon_2)(q-1)}{\tau q}\right)^{1/(q-1)} = \exp\left[\frac{1}{q-1}\ln_e\left(1 + \frac{(\varepsilon_1 - \varepsilon_2)(q-1)}{\tau q}\right)\right]$$
$$\simeq \exp\left[\frac{(\varepsilon_1 - \varepsilon_2)}{\tau q} - \frac{1}{2}\frac{(\varepsilon_1 - \varepsilon_2)^2(q-1)}{\tau^2 q^2}\right] \tag{2.12}$$

we have

$$N_1 = \frac{N\exp[-(\varepsilon_1 - \varepsilon_1)/\tau q + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)^2(q-1)/\tau^2 q^2]}{1 + \exp[-(\varepsilon_1 - \varepsilon_2)/\tau q + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)^2(q-1)/\tau q]}$$
$$N_2 = \frac{N}{1 + e^{\frac{-(\varepsilon_1 - \varepsilon_2)}{\tau q} + \frac{1}{2}\frac{(\varepsilon_1 - \varepsilon_2)^2}{\tau^2 q^2}(q-1)}} \tag{2.13}$$

For the magnetization of N spins $(\varepsilon_1 = \overline{\mu}B,\ \varepsilon_2 = -\overline{\mu}B)$ we have

$$M = N_2\overline{\mu} - N_1\overline{\mu} \tag{2.14}$$

or upon using equations (2.11)–(2.14)

$$M = N\overline{\mu} \tanh\left(\frac{\overline{\mu}B}{\tau q}\right)\left[1 - e^{-\overline{\mu}B/\tau q}\,\frac{4\overline{\mu}^2B^2(q-1)}{\tau^2 q^2}\right.$$
$$\left.\times\left(\frac{1}{\sinh(\overline{\mu}B/\tau q)} + \frac{1}{\cosh(\overline{\mu}B/\tau q)}\right)\right] \qquad (2.15)$$

We see that equation (2.15) gives the first correction, which varies as $(q - 1)$ and B^2 [since $\sinh(\overline{\mu}B/\tau q) \simeq \overline{\mu}B/\tau q$].

Thus the Renyi statistics decreases the effective magnetization. When $q = 1$, both equations (2.5) (Tsallis statistics) and (2.15) (Renyi statistics) give the Boltzmann Gibbs value of M, $M = N\overline{\mu} \tanh(\overline{\mu}B/kT)$ (Lee *et al.*, 1963).

3. CONCLUSION

The above calculations demonstrate that the first correction to M for Tsallis statistics varies as B^3 and $(q - 1)^2$ (increases M), and for Renyi statistics the correction varies as B^2 and $(q - 1)$ (decreases M). The sign and the variation of the correction provide us with a window to search for nonextensive statistics in magnetic systems. In this regard Torres *et al.* (1997) have set a limit for q of $|q - 1| \leq 2 \times 10^{-5}$ based on the primordial helium abundance in the early universe, and multifractal models are capable of calculating q from the fundamental fractal geometry (Gouyet, 1996). Perhaps a combination of these ideas can be used to search for the presence of nonextensive statistics in magnetic systems. Probably even a better place to search for these effects is in ferromagnetic systems where a fractal geometry is intrinsic to the domainlike structure. Lastly, it is interesting that both (2.5) and (2.15) reduce to the Boltzmann Gibbs values for M when $q \to 1$ and the reason that the Renyi correction to the magnetization is not symmetric upon exchanging $B \to -B$ is because of the presence of $\sinh(\overline{\mu}B/\tau q)$ in the denominator of the correction term, which is singular at $B = 0$. Thus $\overline{\mu}B/\tau q \neq 0$ is assumed in the derivation of the correction term.

ACKNOWLEDGMENTS

I like to thank the physics departments of Williams College and Harvard University for the use of their facilities.

REFERENCES

Andrade, R. F. S. (1991). *Physica A,* **175**, 285.
Andrade, R. F. S. (1994). *Physica A,* **203**, 486.
Bahcall, J. N., and Pinsonneault, M. H. (1992). *Reviews of Modern Physics,* **64**, 885.
Chame, A., and de Mello, E. M. L. (1994). *Journal of Physics A,* **27**, 3663.
Clayton, D. B. (1974). *Nature,* **249**, 131.
Gouyet, J. F. (1996). *Physics and Fractal Structure,* Masson, Paris.
Hiley, B. J. and Joyce, G. S. (1965). *Proceedings of the Physical Society,* **85**, 493.
Kaniadakis, G., Lavagno, A., and Quaràti, P. (1996). *Physics Letters B,* **369**, 308.
Lee, J. F., Sears, F. W., and Turcotte, D. L. (1963). *Statistical Thermodynamics,* Addison-Wesley, Reading, Massachusetts.
Mariz, M. (1992). *Physics Letters A,* **165**, 409.
Montroll, E. W., and Schlesinger, M. F. (1983). *Journal of Statistical Physics,* **32**, 209.
Pavon, A. (1987). *General Relativity and Gravitation,* **19**, 375.
Plastino, A. R., and Plastino, A. (1993). *Physics Letters A,* **177**, 177.
Plastino, A. R., Plastino, A., and Tsallis, C. (1994). *Journal of Physics A,* **27**, 5207.
Ramshaw, J. D. (1993a). *Physics Letters A,* **175**, 169.
Ramshaw, J. D. (1993b). *Physics Letters A,* **175**, 171.
Renyi, A. (1970). *Probability Theory,* North-Holland, Amsterdam.
Stariolo, D. A. (1994). *Physics Letters A,* **185**, 262.
Tirnakli, U., Demirhan, D., and Büyükkilic, F. (1997a). *Acta Physica Polonica A,* **91**, 1035.
Tirnakli, U., Büyükkilic, F., and Demirhan, D. (1997b). *Physica A,* in press.
Torres, D. F., Vucetech, H., and Plastino, A. (1997). *Physics Review Letters,* **79**, 1588.
Tsallis, C. (1988). *Journal of Statistical Physics,* **52**, 479.
Tsallis, C., Sa Berreto, F. C., and Loh, E. D. (1995). *Physical Review E,* **52**, 1447.